

RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIFTH SEMESTER EXAMINATION, DECEMBER 2013

THIRD YEAR

MATHEMATICS (Honours)

Date : 21/12/2013

Time : 11 am – 3 pm

Paper : VI

Full Marks : 100

[Use a separate answer book for each group]

Group – A

Unit - I

(Answer any six questions)

1. Let $f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x^2 + y^2 = 0 \end{cases}$
 - a) Examine for the existence of directional derivative at (0,0) along the direction of unit vector (a, b) and
 - b) hence state, with reasons, whether f is differentiable at (0,0)
 - c) Can you infer the existence of f_x and f_y at (0,0) from your result (a)? [1+2+1+1]
2. If $S \subseteq \mathbb{R}^n$, let $f : S \rightarrow \mathbb{R}^m$ be a function with values in $\mathbb{R}^m$, and write $f = (f_1, f_2, \dots, f_m)$. Prove that f is differentiable at an interior point c of S if, and only if, each f_i is differentiable at c. [5]
3. Let f and g be functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Assume that f is differentiable at c, that $f(c) = 0$, and that g is continuous at c. Let $h(x) = g(x) \cdot f(x)$. Prove that h is differentiable at c and that $h'(c)(u) = g(c) \cdot \{f'(c)(u)\}$ if $u \in \mathbb{R}^n$. [5]
4. Let $u(x, y)$ be homogeneous function of degree $m (\neq 0)$, having continuous second order partial derivatives and satisfy Laplace's equation.
Let $v(x, y) = m^{-1}(yu_x - xu_y)$. Prove that v satisfies Laplace's equation. [5]
5. Using the property of Jacobian, show that f, g and h are functionally related where
 $f(x, y, z, w) = x + y + z + w$, $g(x, y, z, w) = x^2 + y^2 + z^2 + w^2$
 $h(x, y, z, w) = 2(xy + yz + zx + xw + yw + zw)$. [5]
6. Given that $u(x, y)$ and $v(x, y)$ are differentiable functions and are not functionally related and
 $\frac{\partial u}{\partial x} = \left\{ \frac{\partial v}{\partial y} \right\}^2$. Show that $\left(\frac{\partial y}{\partial v} \right) \cdot \frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial x}{\partial u} \right)^2$. [5]
7. a) Let f and g be functions on $\mathbb{R}$ with continuous derivatives, $f(0) = 0$, $f'(0) \neq 0$. Consider the equation $f(x) = tg(x)$; $x, t \in \mathbb{R}$. Show that in a suitable interval $|t| < \delta$, there exists a unique continuous function $x = x(t)$ that solves the equation and satisfies $x(0) = 0$. [2]
b) Let $f(p, q, r, s) = p^2 + q^2 - rp + \sin(s + p) = 0$
 $g(p, q, r, s) = p^3q + \cos(p + 2q + r - s) - q - 1 = 0$
Explain whether we can solve for p and q in terms of r and s. [3]
8. Let $f(x, y) = \sqrt{|xy|}$. Show that the Taylor's expansion is not valid about the point (x, x) in any neighbourhood of (x, x) containing (0, 0). [5]

9. Find the lengths of the axes of the ellipse $5x^2 + 8xy + 5y^2 = 9$ using the method of Lagrange's multipliers. [5]

Unit - II

(Answer **any four** questions)

10. a) Prove that $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$ where $\vec{A}$ and $\vec{B}$ are differentiable vector functions of t .
Hence deduce $\frac{d}{dt}(\vec{A} \times (\vec{B} \times \vec{C})) = \frac{d\vec{A}}{dt} \times (\vec{B} \times \vec{C}) + \vec{A} \times \left(\frac{d\vec{B}}{dt} \times \vec{C} \right) + \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{dt} \right)$. [4]
- b) For the circular motion of a particle, $\vec{v} = \vec{\omega} \times \vec{r}$, assuming that $\vec{\omega}$ is constant, $\text{div } \vec{v} =$ [1]
i) $2|\vec{\omega}|$ ii) $|\vec{\omega}|$ iii) $-2|\vec{\omega}|$ iv) 0
11. a) Prove that $\vec{\nabla} \times (\vec{f} \times \vec{g}) = \vec{f}(\vec{\nabla} \cdot \vec{g}) - \vec{g}(\vec{\nabla} \cdot \vec{f}) - (\vec{f} \cdot \vec{\nabla})\vec{g} + (\vec{g} \cdot \vec{\nabla})\vec{f}$ where $\vec{f}$ and $\vec{g}$ are differentiable functions. [3]
- b) If $\vec{a}$ and $\vec{b}$ are irrotational, then $\vec{a} \times \vec{b}$ is [2]
i) irrotational ii) solenoidal
iii) irrotational and solenoidal both iv) neither irrotational nor solenoidal
—Justify your answer.
12. Prove that a field $\vec{F}$ be conservation iff $\vec{F} = \vec{\nabla}\phi$, where ϕ is a scalar differentiable point function. [5]
13. a) Evaluate $\oint_C [(3x+4y)dx + (2x-3y)dy]$ where C , a circle of radius 2 with centre at the origin of the x-y plane, is traversed in the positive sense. [4]
- b) If $\vec{t}$ is the unit tangent vector and C is the unit circle in x-y plane with centre at the origin, then the value of $\oint_C \vec{t} \cdot d\vec{r} =$ [1]
i) 2π ii) π iii) $\pi/2$ iv) 0
14. a) Deduce from divergence theorem $\iint_S \vec{n} \times \vec{F} dS = \iiint_V \vec{\nabla} \times \vec{F} dv$. [2]
- b) Use Gauss' Divergence Theorem to evaluate $\iint_S (4xz\vec{i} - y^2\vec{j} + 4yz\vec{k}) \cdot \vec{n} dS$, where S is the surface of the solid bounded by the sphere $x^2 + y^2 + z^2 = 10$ and the paraboloid $x^2 + y^2 = z - 2$. [3]
15. Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$. [5]

Group – B

Answer **any two** questions from Question No. 16-18 :

16. a) Find the conditions that must be satisfied for a given line to be a principal axis of a material system at any point on its length. Find the other two principal axes. [7]
- b) Two equal uniform rods AB and AC are freely jointed at A. They are placed on a table so as to be at right angles. The rod AC is struck by a blow at C in a direction perpendicular to AC. Show that the resulting velocities of the middle points of AB and AC are in the ratio 2 : 7. [8]
17. a) Define a compound pendulum.

A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\frac{h}{5}(4 + \tan^2 \alpha)$. [8]

b) A uniform sphere of radius 'a' rolls down an inclined plane, rough enough to prevent any sliding. Discuss the motion and prove that for pure rolling $\mu > \frac{2}{7} \tan \alpha$, where α is the inclination of the plane to the horizon and μ is the co-efficient of friction. [7]

18. a) Prove that the kinetic energy of a rigid body of mass M moving in two dimensions is equal to the sum of the kinetic energy of the particle of mass M placed at the centre of inertia and moving with it and the kinetic energy of the body relative to the centre of inertia. [6]

b) A uniform rod of length $2a$, is placed with one end in contact with a perfectly rough horizontal table and is then at an inclination α to the horizon and is allowed to fall. When it becomes horizontal, show that the angular velocity is $\sqrt{\frac{3g}{2a} \sin \alpha}$. [6]

c) Explain what is meant by the length of a 'simple equivalent pendulum'. [3]

Answer **any two** questions from Question No. 19-21 :

19. A comet is moving in a parabola about the Sun as focus; when at the end of its latus rectum its velocity suddenly becomes altered in the ratio $n:1$, where $n < 1$; show that the comet will describe an ellipse whose eccentricity is $\sqrt{1 - 2n^2 + 2n^4}$ and whose major axis is $\frac{\ell}{1 - n^2}$, where 2ℓ is the latus rectum of the parabolic path. [7]

20. A spherical raindrop of radius 'a', falling freely, receives in each instant an increase of volume equal to λ times its surface at that instant. Show that the velocity at the end of time t is given by $\frac{g}{4\lambda} \left\{ a + \lambda t - \frac{a^4}{(a + \lambda t)^3} \right\}$. [7]

21. The base of a rough cycloidal arc is horizontal, its axis vertical and vertex downwards. A bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that $\mu^2 e^{\mu\pi} = 1$, μ being the co-efficient of friction. [7]

Answer **one** question from Question No. 22-23 :

22. When a particle is describing an elliptic orbit, determine the effect on the orbit of an instantaneous change in the value of the absolute acceleration μ . [6]

23. A particle is projected along the inner surface of a rough sphere and is acted on by no forces. Show that it will return to the point of projection at the end of time $\frac{a}{\mu V} (e^{2\pi\mu} - 1)$, where a is the radius of the sphere, V is the velocity of projection and μ is the co-efficient of friction. [6]

